PROBLEM ON 2009 OCTOBER 19

MVHS NUMBER THEORY GROUP

Last Wednesday we characterized all integers divisible by 9. We did this by writing the number with digits

$$d_n d_{n-1} \dots d_2 d_1 d_0$$

as a polynomial in the base 10 as follows.

$$d_n d_{n-1} \dots d_2 d_1 d_0 = d_n \cdot 10^n + d_{n-1} \cdot 10^{n-1} + \cdots + d_2 \cdot 10^2 + d_1 \cdot 10 + d_0$$

For example

$$341 = 3 \cdot 10^2 + 4 \cdot 10 + 1$$

We then realized that we could rewrite

$$\begin{aligned} d_n d_{n-1} \dots d_2 d_1 d_0 &= d_n \cdot 10^n + d_{n-1} \cdot 10^{n-1} + \cdots + d_2 \cdot 10^2 + d_1 \cdot 10 + d_0 \\ &= d_n \cdot (10^n - 1) + d_{n-1} \cdot (10^{n-1} - 1) + \cdots + d_2 \cdot 99 + d_1 \cdot 9 + (d_n + d_{n-1} + \cdots d_2 + d_1 + d_0) \end{aligned}$$

and split the terms into two groups, namely

$$\underbrace{d_n \cdot (10^n - 1) + d_{n-1} \cdot (10^{n-1} - 1) + \cdots + d_2 \cdot 99 + d_1 \cdot 9}_{\text{Term 1}} + \underbrace{(d_n + d_{n-1} + \cdots d_2 + d_1 + d_0)}_{\text{Term 2}}$$

Term 1 is divisible by 9 since $10^i - 1$ is a string of i nines for any positive integer i, and can be written as

$$\begin{aligned} \overbrace{99 \cdots 9}^{i \text{ times}} &= 9 \cdot 10^i + 9 \cdot 10^{i-1} + \cdots + 9 \cdot 10^2 + 9 \cdot 10 + 9 \\ &= 9 \cdot \left(10^i + 10^{i-1} + \cdot + 10^2 + 10 + 1\right) \end{aligned}$$

Thus, in order for our original number $d_n d_{n-1} \dots d_2 d_1 d_0$ to be divisible by 9, Term 2 must also be divisible by 9. Also if Term 2 is divisible by 9, then our original number is divisible by 9. But what is Term 2? Term 2 is the sum of the digits of the number. For example,

$$123456789$$

is divisible by 9 and every reordering of the digits is divisible by 9. On Wednesday we also showed how to characterize all numbers divisible by 11. Can you find a similar way to describe *all* numbers divisible by 7 and explain your solution? If so, this problem is worth **2 Points**.